Tying Up Some Loose Ends on Fundamentals

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Lecture Objectives

• Relate wavelength of radiation to photon energy
• Recognize components of the electromagnetic spectrum
• Calculate relativistic energies
• Express rest mass in terms of energy
• Identify half-life, average life

Quantum nature of radiation

• Dual nature of photon radiation – energy wave with particle properties
  – Discrete energy values

\[ E = h \nu = \frac{hc}{\lambda} \]

• \( E \) = photon energy
• \( h \) = Planck’s constant = \( 6.63 \times 10^{-34} \) J s
• \( \nu \) = frequency of the wave [s\(^{-1}\)]
• \( c \) = speed of light = \( 3.00 \times 10^8 \) m s\(^{-1}\) (2.9979)
Quantum nature of radiation

- Energy often expressed in eV
  - $1 \text{ eV} = $ energy attained by an electron accelerated through a potential of $1 \text{ V}$
- Wavelength often expressed in Angstrom units ($\text{Å}$)
  - $1 \text{ Å} = 10^{-10} \text{ m}$

\[ E = \frac{hc}{\lambda} \]
\[ = \frac{6.63 \times 10^{-34} [\text{J} \cdot \text{s}] \times 3.00 \times 10^8 [\text{m} \cdot \text{s}^{-1}]}{1.989 \times 10^{-35} [\text{J} \cdot \text{m}]} \]
\[ = \frac{1.989 \times 10^{-35} [\text{J} \cdot \text{m}]}{1.6 \times 10^{-19} [\text{J} \cdot \text{eV}^{-1}]} \]
\[ = 1.24 \times 10^{-4} [\text{J} \cdot \text{Å}^{-1}] \]

Energy conventions

- $\text{eV, keV, MeV}$ – used to express energy
  - Used for electron energies (nearly mono-energetic)
  - Used for mono-energetic photons
- $\text{kVp, MVp}$ – used to express peak energy
  - Used for polyenergetic photon spectrum
- $\text{V, kV, MV}$ – used to express voltage
  - Used for accelerating potential in x-ray source

In an 18 MV linac, electrons are accelerated to 18 MeV (approximately)
- Can be removed from the accelerator as an 18 MeV electron beam
- Can be directed onto a target and generate an 18 MV x-ray beam (polyenergetic spectrum)
- The energy of the photons from a $^{60}\text{Co}$ source is 1.25 MeV (mono-energetic)
### Electromagnetic spectrum

- **Radio waves** – speak of frequency
  - AM: 550 kHz – 1500 kHz
  - FM: 87.9 MHz – 107.9 MHz
- **Microwaves** – speak of frequency (MHz), wavelength (cm)
  - 450 MHz – hyperthermia
  - 3000 MHz – linac power supply
    * $\lambda = 1$ cm (cross-section dimension of linac waveguide)

### Electromagnetic spectrum

- **Infrared, visible, ultraviolet** – speak of wavelength in Angstrom range
- **X-rays**
  - Diagnostic imaging
    - Mammography: $30 < kV < 40$
    - Conventional: $70 < kV < 140$
    - CT scanner: kV ~ 120
  - Therapy
    - Grenz ray: $< 20$ kV
    - Contact: $20 < kV < 50$
    - Superficial: $50 < kV < 150$
    - Orthovoltage: $150 < kV < 300$ (practical limit)
    - Megavoltage: $1 < MeV < 50$

### Relativistic energy

- **Classical**: $KE = \frac{1}{2}mv^2$
- **Relativistic**: As electron increase energy, speed approaches $c$, so increase in energy comes from increase of mass
- Energy given by: $E = mc^2$
- Rest energy: $m_0c^2$
  - Mass increases with velocity $\beta = v/c$
Relativistic energy

- Mass increases with velocity $\beta = \frac{v}{c}$

$$m = \frac{m_0}{\sqrt{1 - \beta^2}}$$

$$KE = \frac{m_0c^2}{\sqrt{1 - \beta^2}} - m_0c^2 = m_0c^2\left(\frac{1}{\sqrt{1 - \beta^2}} - 1\right)$$

Total energy = $mc^2 = \frac{m_0c^2}{\sqrt{1 - \beta^2}}$

- Note that in nonrelativistic limit (small $\beta$)

$$\left(\frac{1}{\sqrt{1 - \beta^2}} - 1\right) \approx \left(1 + \frac{1}{2} \beta^2 - 1\right)$$

$$= \frac{1}{2} \beta^2$$

$$KE = m_0c^2 \frac{1}{2} \frac{\nu^2}{c^2}$$

$$= \frac{1}{2} m_0 \nu^2$$

$\beta$ and $m$ for various energies

<table>
<thead>
<tr>
<th>Energy</th>
<th>$\beta = \frac{v}{c}$</th>
<th>$\frac{m}{m_0}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>electron</td>
<td>100 keV 0.5</td>
<td>1.2</td>
</tr>
<tr>
<td></td>
<td>1 MeV 0.94</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>10 MeV 0.999</td>
<td>20.5</td>
</tr>
<tr>
<td>proton</td>
<td>100 MeV 0.43</td>
<td>1.11</td>
</tr>
</tbody>
</table>

- Note that electrons used in radiation therapy are relativistic
Important energy/mass relationships

• Rest mass of electron = 0.511 MeV
• 1 amu = 931.5 MeV

Exponential behavior

• If a quantity changes by a certain factor in a given interval, behavior is exponential
  – Cell growth
  – Cell kill
  – Radioactive decay
  – Buildup of radioactive material
  – Attenuation of photons

Exponential behavior

• All of these are stochastic events – controlled by the laws of probability
• Suppose we have $n$ nuclei of a radioactive material
• What is the probability that $x$ of these will decay assuming that the probability that any one nucleus will decay is $p$?
Exponential behavior

- Each decay event is an independent event, so the probability of any one event occurring is given by the binomial distribution

\[ P(x; n, p) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} \]

Exponential behavior

- Assume that the number of radioactive nuclei is very large, as compared to the expected number of events, i.e., \( n \gg 1 \).

- Write

\[ P(x; n, p) = \frac{1}{n!} \frac{n!}{(n-x)!} p^x (1-p)^{n-x} (1-p)^{1-x} \]

Exponential behavior

- Expand the second term:

\[ \frac{n!}{(n-x)!} = n(n-1)(n-2)\cdots(n-x+2)(n-x+1) \]

- Because \( n \gg x \), all factors in the second term are approximately equal, consequently

\[ \frac{n!}{(n-x)!} \approx n^x \]
Exponential behavior

- In terms of the expectation value of the number of decays \( n = \mu / p \)

\[
P(x; n, p) \approx \frac{1}{x!} \mu^x (1 - p)^x (1 - p)^n
\]

- The next term \((1 - p)^x\) can be written as \((1 + px)\), which is approximately 1 because both \( p \) and \( x \) are small

Exponential behavior

- Writing \( n = \mu / p \) and taking the limit as \( p \to 0 \),

\[
\lim_{p \to 0} (1 - p)^n = \lim_{p \to 0} [(1 - p)^\mu]^\frac{\mu}{p} = \left( \frac{1}{e} \right)^\mu = e^{-\mu}
\]

Exponential behavior

- Consequently

\[
\lim_{p \to 0} [P(x; n, p)] = \frac{e^{-\mu}}{x!} \mu^x
\]

\[
= \frac{e^{-\mu}}{x!} e^{x \log \mu}
\]
Exponential behavior

- The quantity $\mu$ is a deterministic quantity, which represents the mean value of the stochastic variable $x$ (number of decay events).
- Because the decay events occur over a specified time interval, we can write $\mu = \lambda t$.
- The quantity $\lambda$ can then be interpreted as the mean number of decay events per unit time.

Exponential behavior

- We want to determine how are the decay events distributed over time.
- How many nuclei are present at a specified time, i.e., how many have not decayed.
- What is the probability that an event has not happened, that is, what is the probability that $x=0$.

Exponential behavior

- Consequently

$$\lim_{p \to 0, n \to 0} P(x; n, p) = \frac{e^{-x}}{x!} e^{\lambda t}$$

$$= e^{-\lambda t}$$
Exponential behavior

- The expected number of radioactive nuclei present is equal to the original number present multiplied by the probability that no interaction has occurred

\[ N = N_0 e^{-\lambda t} \]

That was the stochastic approach

- Exponential behavior is the non-stochastic average of a stochastic event

\[ \Delta N = \pm \lambda N \Delta t \]

- \( \Delta N \): quantity
- \( \pm \): + for growth, - for reduction
- \( \lambda \): proportionality constant

Non-stochastic representation – look at differential expression

\[ dN = \pm \lambda N \, dt \]

or \[ dN/N = \pm \lambda \, dt \]

- Integrating, we get

\[ N = N_0 e^{\pm \lambda t} \]
Exponential behavior

- Special circumstances:
  - $N = \frac{1}{2}N_0$
  - $t = t_{\frac{1}{2}}$ – the interval to reduce $N$ to half its initial value

\[
\frac{N}{N_0} = e^{-\lambda t}
\]

\[
\log\left(\frac{N}{N_0}\right) = -\lambda t
\]

\[
\log(0.5) = -\lambda t_{\frac{1}{2}}
\]

\[
t_{\frac{1}{2}} = \frac{\log 2}{\lambda} = 0.693
\]

Exponential behavior

- Special circumstances:
  - $N = N_0/e$
  - $t = t_a$

\[
\log\left(\frac{N_0/e}{N_0}\right) = \log(1/e) = -\lambda t_a
\]

\[
\log(e) = 1 = 2\lambda t_a
\]

\[
t_a = \frac{1}{\lambda}
\]

Plotting exponential behavior
Plotting exponential behavior

![Graph showing exponential behavior](image)