Lecture Objectives

- Define radioactive equilibrium and identify when it occurs
- Differentiate between transient and secular equilibrium
- Perform calculations related to activation of radioactive isotopes

Radiation equilibrium

- If daughter has half-life longer than that of parent, daughter activity increases as parent activity decays. When parent gone, amount of daughter decreases based on daughter’s own half-life. – Trivial case
Radiation equilibrium

- If daughter has half-life shorter than that of parent, daughter is produced continuously, but decays. When rate of decay equals rate of production, daughter appears to decay at same rate as parent. – Radiation equilibrium

\[
\frac{dN_d}{dt} = -\frac{dN_p}{dt} - \lambda_d N_d
\]

\[
= \lambda_p N_0 e^{-\lambda_p t} - \lambda_d N_d
\]

Radiation equilibrium

- Rate of change of daughter = creation by parent minus decay of daughter

\[
N_d(t) = \left( \frac{\lambda_p}{\lambda_d - \lambda_p} \right) N_0 \left( e^{-\lambda_p t} - e^{-\lambda_d t} \right)
\]

Radiation equilibrium

- Initial conditions (t=0)
  - \( N_p = N_0 \)
  - \( N_d = 0 \)

- Solution of differential equation is

\[
N_d(t) = \left( \frac{\lambda_p}{\lambda_d - \lambda_p} \right) N_0 \left( e^{-\lambda_p t} - e^{-\lambda_d t} \right)
\]
Radiation equilibrium

• Factor out

\[ A_p(t) = \lambda_p N_0 e^{-\lambda_p t} \]

to obtain

\[ N_d = A_p(t) \left( \frac{1}{\lambda_d - \lambda_p} \right) \left[ 1 - e^{-(\lambda_d - \lambda_p) t} \right] \]

Radiation equilibrium

• Recall that activity \( A = \lambda N \) and at \( t=0 \)
  – \( A_p = A_0 \)
  – \( A_d = 0 \)

\[ A_d = A_p(t) \left( \frac{\lambda_d}{\lambda_d - \lambda_p} \right) \left[ 1 - e^{-(\lambda_d - \lambda_p) t} \right] \]

Transient equilibrium

• Half-life of parent is longer than half-life of daughter: \( \lambda_d > \lambda_p \)
  – At large \( t \), neglect exponential term

\[ A_d(t) = A_p(t) \left( \frac{\lambda_d}{\lambda_d - \lambda_p} \right) \]
Transient equilibrium

- Daughter decays at same rate as it is produced by the parent
- Daughter appears to decay with half-life of parent
- Activity of daughter is greater than activity of parent

\[ A_d = A_p \left( \frac{\lambda_d}{\lambda_d - \lambda_p} \right) \]

Transient equilibrium

- Maximum value of daughter activity occurs at time \( t_{\text{max}} \)
  - At this time, \( A_d = A_p \)
- Optimum time to extract daughter is at \( t_{\text{max}} \)

\[ t_{\text{max}} = \frac{\log(\lambda_d/\lambda_p)}{\lambda_d - \lambda_p} \]
Secular equilibrium

- Half-life of parent is very much longer than half-life of daughter - $\lambda_d >> \lambda_p$
  - Neglect $\lambda_p$ subtracted from $\lambda_d$

$$A_d = A_p \left(1 - e^{-\lambda_d t}\right)$$

Use of radon

Radiation equilibrium

- Things get a bit messier if the parent does not decay to the given daughter 100% of the time (a frequent occurrence)
  - Still use equilibrium terminology

Activation of isotopes

- Result is similar to production of daughter when parent half-life is much longer than daughter’s
- Put a mass of an element into reactor or accelerator
- Probability of interaction with element proportional to interaction cross section $\sigma$

Activation of isotopes

- Conceptually, $\sigma$ is the cross-sectional area of the nucleus as it appears to the bombarding particle \(\iff\) area around the nucleus which, if the bombarding particle enters, the interaction will occur
Activation of isotopes

• Rate of production of created isotope given by
  \[ \frac{dN}{dt} = N_t \sigma \phi \]
  - \( N_t \): number of atoms of the target element present
  - \( \phi \): neutron fluence rate [neutrons cm\(^{-2}\) s\(^{-1}\)]
  - \( \sigma \): cross section [cm\(^2\) neutron\(^{-1}\)]

Example: Place 1 g \(^{59}\)Co in reactor with neutron fluence rate of \(10^{13}\) n cm\(^{-2}\) s\(^{-1}\) for 1 yr. (Ignore decay of \(^{60}\)Co.)

\[
N_{Co-60} = N_{Co-59} \cdot \sigma \cdot \phi \cdot t \\
= 1[g] \cdot 6.023 \times 10^{23}[\text{atoms/mole}] \cdot 37 \times 10^{-24}[\text{cm}^{-3}] \\
\cdot 59[\text{g/mole}] \\
\cdot 10^{13}[\text{cm}^{-2} \text{s}^{-1}] \cdot 1[\text{yr}] \cdot 365 \cdot 24 \cdot 60 \cdot 60[\text{sec}^{-1}] \\
= 1.19 \times 10^{20} \text{ [atoms]}
\]
Activation of isotopes

- The half-life of $^{60}\text{Co}$ is 5.26 yr. What is the activity?

\[
A = \lambda N = \frac{0.693}{5.26 \text{ yr} \cdot 365 \cdot 24 \cdot 60 \text{ s yr}^{-1}} \cdot 1.19 \times 10^{20} \\
= 4.97 \times 10^{10} \text{ Bq} = \frac{4.97 \times 10^{10} \text{ Bq}}{3.7 \times 10^9 \text{ Bq/Ci}} \\
= 1.34 \text{ Ci}
\]

Activation of isotopes

- Note that 1 g of $^{59}\text{Co}$ in a neutron fluence of $10^{13}$ [neutrons cm$^{-2}$ s$^{-1}$] produces 1 Ci per year. A radiation therapy unit needs approximately 5 kCi.

- This is an oversimplification
  - Does not include decay of $^{60}\text{Co}$ (significant oversimplification)
  - Assumes constant fluence rate (approximately average rate)
  - Neglects shielding of one atom by another (may be important)
  - No change in $N_t$ (not bad – approx 1% effect)

Activation of isotopes

- Add decay to equation:

\[
\frac{dN}{dt} = N_t \cdot \sigma \cdot \phi - \lambda N
\]

- Eventually production equals decay
  - $dN/dt = 0$ and we obtain saturation

\[
N_t \cdot \sigma \cdot \phi = \lambda N_{\text{sat}}
\]

\[
A_{\text{sat}} = N_t \cdot \sigma \cdot \phi
\]
Activation of isotopes

• The solution to the differential equation is

\[ A = A_0 \left(1 - e^{-\lambda t}\right) \text{ where} \]
\[ A_0 = N_i \cdot \sigma \cdot \phi \]

Radioactive implants

• Dose from isotopes left in place for significant period of time – activity may change significantly during implant

Radioactive implants

• Let \( \frac{dD}{dt} \) be dose rate at an appropriate point in the patient.

Then

\[
D = \int \frac{dD}{dt} \, dt
\]
\[
\frac{dD}{dt} = \frac{dN_i}{dt} e^{-\lambda t}
\]
\[
D = \int \frac{dN_i}{dt} \, e^{-\lambda t} dt = \frac{dN_i}{dt} \left(1 - e^{-\lambda t}\right)
\]
\[
= 1.44t \frac{dN_i}{dt} \left(1 - e^{-\lambda t}\right)
\]
Radioactive implants

• For an implant left in permanently
  \( t \to \infty \), so

\[
D = 1.44t^{1/2} \frac{dD_0}{dt}
\]