Interactions of Photons with Matter – Compton Scatter (Part 1)

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Lecture Objectives

• Identify and describe the Compton scatter process
• Calculate the attenuation coefficient for Compton scatter
• Determine the dependence of the attenuation coefficient on nature of the absorber and on photon energy

Importance of process

• Most important interaction in radiation oncology
  – In soft tissue dominant interaction in energy range 30 keV – 30 MeV
Description of process

- Kinematics
  - Relates energies and angles of particles
- Cross section
  - Predicts probability that an interaction will occur

Description of process – Follow the energy

- Photon interacts with single electron
  - Inelastic scatter: loss of energy
  - Some energy given to electron
  - Some energy retained by scattered photon
  - Energetic electron ejected from atom
  - Causes further ionization – We’ll learn about this later
  - May cause biological damage
  - Scattered photon causes further ionization

Description of process

[Diagram showing incident photon, ejected electron, scattered photon, and energy relations]
Kinematics – follow energy and momentum

• First approximation
  – Free electron (Why?)
  – Neglect binding energy
• Conserve energy and momentum

\[ p = p' \cos \theta + q \cos \phi \]
\[ p' \sin \theta = q \sin \phi \]
\[ h \nu = h \nu' + E_e \]

Kinematics

• Photon momentum \( p = h \nu / c \)
• Need relativistic mechanics for energy-momentum relationship for electron

\[
q = \frac{m_e v}{\sqrt{1 - \beta^2}} \quad \left( \beta = \frac{v}{c} \right)
\]

\[ E_e = m_e c^2 \left[ \frac{1}{\sqrt{1 - \beta^2}} - 1 \right] \]

kinetic energy of \( e^- \) = total energy - rest mass energy
Kinematics

- Solving equations, we find

\[ h\nu' = h\nu \frac{1}{1 + \alpha(1 - \cos \theta)} \]
\[ E_e' = h\nu \frac{\alpha(1 - \cos \theta)}{1 + \alpha(1 - \cos \theta)} \]
\[ \alpha = \frac{h\nu}{m_e c^2} = \frac{h\nu}{0.511} \]
\[ h\nu = h\nu + E_e' \]

Kinematics

- For incident photon energy, \( h\nu \), energy of scattered photon and energy of Compton electron are related to angle of scattered photon \( \theta \).

Energy change

- Note: Change in wavelength of photon is independent of energy, function only of scatter angle

\[ \lambda' - \lambda = \frac{hc}{m_e c^2}(1 - \cos \theta) \]
\[ \Delta \lambda(\lambda) = 0.0243(1 - \cos \theta) \]
Limits

• Direct hit:
  – Electron scattered in forward direction ($\phi = 0$)
  – Photon scattered backward ($\theta = \pi$)

$$\begin{align*}
  h\nu^\prime &= h\nu \frac{1}{1 + \alpha(1 - \cos \theta)} \\
  E_e^\prime &= h\nu \frac{\alpha(1 - \cos \theta)}{1 + \alpha(1 - \cos \theta)}
\end{align*}$$

$$\begin{align*}
  h\nu_{\text{min}} &= h\nu \frac{1}{1 + 2\alpha} \\
  E_{e\text{ max}} &= h\nu \frac{2\alpha}{1 + 2\alpha}
\end{align*}$$

• As $\alpha$ (energy of incident photon) increases, the energy of the Compton electron increases and approaches $h\nu$.
• As $\alpha$ decreases, the energy of the Compton electron approaches 0.
• This is true irrespective of angle of scatter.
Limits

\[
\begin{align*}
\nu'_{\text{min}} &= \nu \frac{1}{1 + 2\alpha} \\
E'_{\text{max}} &= \nu \frac{2\alpha}{1 + 2\alpha}
\end{align*}
\]

- Compton scatter is poor energy transfer mechanism at low energies but becomes much better at higher energies.

Limits

- Look at situation where photon is scattered \( \pi/2 \) (right angle scatter):
  \[
  1 - \cos \theta = 1.
  \]

  \[
  \begin{align*}
  \nu' &= \nu \frac{1}{1 + \alpha (1 - \cos \theta)} \\
  E' &= \nu \frac{\alpha (1 - \cos \theta)}{1 + \alpha (1 - \cos \theta)}
  \end{align*}
  \]

  \[
  \begin{align*}
  \nu'(90^\circ) &= \nu \frac{1}{1 + \alpha} \\
  E'(90^\circ) &= \nu \frac{\alpha}{1 + \alpha}
  \end{align*}
  \]

Limits

- Grazing hit:
  - Photon essentially unscattered (\( \theta = 0 \))
  - Photon energy essentially unchanged.
  - Electron takes minimal energy
  - Electron may not be ejected at more than (\( \phi = \pi/2 \))
    - To conserve momentum and energy, electron must have no momentum in negative x-direction.
Some examples – energy dependence

- Photon energy = 5.11 MeV ($\alpha = 10.0$)
  \[
  \begin{align*}
  \theta = 180^\circ & \quad \frac{E_e (180^\circ)}{h} = \frac{2\alpha}{1+2\alpha} \frac{20}{21} h = 0.95h \\
  \theta = 90^\circ & \quad \frac{E_e (90^\circ)}{h} = \frac{\alpha}{1+\alpha} \frac{10}{11} h = 0.909h
  \end{align*}
  \]
- Electron receives > 90% of incident photon energy – efficient energy transfer

- Lower photon energy = 0.511 MeV ($\alpha = 1.0$)
  \[
  \begin{align*}
  \theta = 180^\circ & \quad \frac{E_e (180^\circ)}{h} = \frac{2\alpha}{1+2\alpha} \frac{2}{3} h = 0.67h \\
  \theta = 90^\circ & \quad \frac{E_e (90^\circ)}{h} = \frac{\alpha}{1+\alpha} \frac{1}{2} h = 0.5h
  \end{align*}
  \]
- Electron receives 50% - 67% of incident photon energy – much less efficient energy transfer

- Very low energy = 5.11 keV ($\alpha = 0.01$)
  \[
  \begin{align*}
  \theta = 180^\circ & \quad \frac{E_e (180^\circ)}{h} = \frac{2\alpha}{1+2\alpha} \frac{0.02}{1+0.02} h = 0.0196h \\
  \theta = 90^\circ & \quad \frac{E_e (90^\circ)}{h} = \frac{\alpha}{1+\alpha} \frac{0.01}{1+0.01} h = 0.0099h
  \end{align*}
  \]
- Scattered photon retains 98-99% of incident photon energy – poor energy transfer
Some examples – energy dependence

- Zero energy limit
  - No energy transferred
  - Classical limit

Some examples

- High energy limit ($\alpha \gg 1$)
  - Consider 90° and 180° scatter
  - Photons scattered back from primary barrier in treatment room

Some examples

\[
\begin{align*}
\theta &= 180^\circ \quad h\nu'(180^\circ) = h\nu \frac{1}{1 + 2\alpha} \approx h\nu \frac{1}{2\alpha} \\
&\text{but } \alpha = \frac{hv}{m_0c^2} \\
&h\nu'(180^\circ) \approx \frac{m_0c^2}{2} \\
\theta &= 90^\circ \quad h\nu'(90^\circ) = h\nu \frac{1}{1 + \alpha} \approx h\nu \frac{1}{\alpha} = m_0c^2
\end{align*}
\]
Some examples

- Photons scattered more than 90°, irrespective of energy, never have energies greater than 0.511 MeV
  - Significant implications for radiation shielding
  - Shielding for primary and leakage photons (much higher energies) is generally adequate to shield for scattered photons

Differential scatter cross section

- We have looked at energy transfer in Compton scatter
- Now look at probability of scatter as function of angle – differential scattering cross section
- Requires quantum mechanical treatment

Klein-Nishina coefficient

- First approximation is that electrons are free electrons – obtain electronic cross section
  \[
  \frac{d\sigma}{d\Omega} = \frac{d\sigma_0}{d\Omega} \cdot F_{KN} = \frac{r_e^2}{2} \left(1 + \cos^2 \theta \right) \cdot F_{KN}
  \]
  = classical differential cross section
  \(\times\) Klein-Nishina coeff
Klein-Nishina coefficient

\[ F_{KN} = \left( \frac{1}{1 + \alpha(1 - \cos \theta)} \right)^2 \left( \frac{\alpha^2 (1 - \cos \theta)^2}{1 + \alpha(1 - \cos \theta) + \alpha^2 \cos^2 \theta} \right) \]

- \( F_{KN} < 1 \) so scatter reduced from that of classical treatment

Some limits

- Low energy: \( \alpha \ll 1 \)
  - \( F_{KN} \approx 1 \) reduces to classical equation

- Small angle scatter: \( \theta = 0, \cos \theta = 1 \)
  - \( F_{KN} \approx 1 \) reduces to classical equation

Some limits

- High energy: \( \alpha \gg 1 \)
  - \( \theta = \pi, \cos \theta = -1 \)

\[ F_{KN} \approx \left( \frac{1}{1 + 2\alpha} \right) \left( 1 + \frac{4\alpha^2}{1 + 2\alpha} \right) \approx \left( \frac{1}{2\alpha} \right)^2 \frac{4\alpha^2}{4\alpha} = 1 \]
Some limits

\[ F_{K\gamma} = \left\{ \frac{1}{1 + \alpha(1 - \cos \theta)} \right\} \left\{ \frac{\alpha^2 (1 - \cos \theta)^2}{1 + \alpha(1 - \cos \theta)(1 + \cos^2 \theta)} \right\} \]

- High energy: \( \alpha \gg 1 \)
- \( \theta = \pi/2, \cos \theta = 0 \)

\[ F_{K\gamma} = \frac{1}{\alpha} \]

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Some limits

\[ F_{K\gamma} = \left\{ \frac{1}{1 + \alpha(1 - \cos \theta)} \right\} \left\{ \frac{\alpha^2 (1 - \cos \theta)^2}{1 + \alpha(1 - \cos \theta)(1 + \cos^2 \theta)} \right\} \]

- \( \theta = 0, \cos \theta = 1 \)

\[ F_{K\gamma} = 1 \text{ (independent of energy)} \]

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Summary

Differential scattering cross section for Compton vs \( \theta \) for various energies

- \( h\nu = 0 \) (classical)
- \( h\nu = 1 \text{ MeV} \)
- \( h\nu = 10 \text{ MeV} \)
Summary: Compton scatter cross sections

- Equal to classical scatter at all angles at zero energy
- Equal to classical scatter at $\theta = 0$
- Peaked in the forward direction as photon energy increases
- Compton scatter at $\theta = \pi$ is lower than that at $\theta = 0$ by
  - About 1 order of magnitude at 1 MeV ($\alpha=2$)
  - About 2 orders of magnitude at 10 MeV ($\alpha=20$)